

# NP-hardness of hypercube 2-segmentation

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## Abstract

The *hypercube 2-segmentation* problem is a certain biclustering problem that was previously claimed to be NP-hard, but for which there does not appear to be a publicly available proof of NP-hardness. This manuscript provides such a proof.

## 1 Introduction

We consider the following problem.

**Hypercube 2-segmentation (H2S).** Given a set of  $k$  vectors  $x_1, \dots, x_k$  in  $\{0, 1\}^d$ , one needs to select two centers  $c_1, c_2$  in  $\{0, 1\}^d$  maximizing

$$\sum_{i=1}^k \max[agree(c_1, x_i), agree(c_2, x_i)]$$

where  $agree(x, y)$  counts on how many coordinates vectors  $x$  and  $y$  agree (which is  $d$  minus the Hamming distance between  $x$  and  $y$ ).

H2S may also be phrased in the following equivalent way.

**H2S –  $\ell_1$  maximization formulation.** Given a set of  $k$  vectors  $x_1, \dots, x_k$  in  $\{1, -1\}^d$ , partition the  $k$  vectors into two sets, such that the sum of the  $\ell_1$  norms of the two corresponding vector sums is maximized.

The equivalence between the two formulations of H2S follows from the fact that for a set of vectors on the hypercube, the location of the center that maximizes agreement is determined by taking the majority value on each coordinate separately. The  $\ell_1$  norm of the sum shows how much this

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optimal center gains compared to placing a center at  $0^d$  (the center of the  $\{1, -1\}^d$  hypercube).

The following theorem was claimed in [3] without proof.

**Theorem 1** *The hypercube 2-segmentation problem is NP-hard.*

In this manuscript we provide a proof of this theorem.

## 1.1 Related work

Kleinberg, Papadimitriou and Raghavan [3] undertook a systematic study of the complexity of segmentation problems. Item (4) of Theorem 2.1 in [3] claims that H2S is NP-hard. The proof sketch of that theorem states that the proof is by reduction “from *maximum satisfiability* with clauses that are equations modulo two”, and gives no further details. That paper also shows that given any set of vectors, at least one of these vectors is a nearly optimal center for the set: its agreement score with the set of vectors is at least a  $2\sqrt{2} - 2 \simeq 0.828$  fraction of that of the optimal center. This easily implies a polynomial time algorithm for approximating H2C within a ratio 0.828.

Alon and Sudakov [1] provide a PTAS for H2S. Specifically, for every choice of  $\epsilon > 0$  they provide a linear time algorithm (with leading constant that depends on  $\epsilon$ ) that approximates H2S with a factor no worse than  $1 - \epsilon$ . Similar results apply to hypergraph  $k$ -segmentation for constant  $k > 2$ .

In [4], the journal version of [3], Item (3) of Theorem 1.1 claims that H2S is MAXSNP-complete, and cites [3] as reference, without providing a proof. This claim of MAXSNP-completeness contradicts the fact (proved in [1]) that H2S has a PTAS, and hence is incorrect. (The authors of [4] do cite [1].)

Wulff, Urner and Ben-David [5] study a problem that they refer to as *monochromatic biclustering* (MCBC). They present a PTAS for (the maximization version of) MCBC, and also prove NP-hardness of MCBC in the case that the input instance may contain *don't care* symbols. This NP-hardness result is based on a reduction from *max-cut*. In [5], it is conjectured that NP-hardness holds even without *don't care* symbols. H2C is a special case of MCBC without *don't care* symbols, and hence Theorem 1 implies the conjecture of [5]. (Apparently, the authors of [5] were unaware of the previous work on H2C cited above. The term *biclustering* does not appear in [4], whereas the term *segmentation* does not appear in [5].)

## 2 Proof of NP-hardness

We start with some background. A Hadamard code  $H_M$  of dimension  $M$  is a collection of  $M$  vectors in  $\{1, -1\}^M$  with the property that every two vectors are orthogonal. There are well known recursive constructions of Hadamard codes when  $M$  is a power of 2, and hence we shall assume  $M$  to be a power of 2.

Recall the notions of  $\ell_1$  and  $\ell_2$  norms of a vector. We shall use the following proposition.

**Proposition 2** *Consider an arbitrary set of distinct vectors from an arbitrary Hadamard code  $H_M$ . Then the  $\ell_1$  norm of their sum is at most  $M^{3/2}$ .*

**Proof.** The  $\ell_2$  norm of a code word is  $\sqrt{M}$ . As the codewords are orthogonal, the  $\ell_2$  norm of the sum of  $q$  distinct vectors is  $\sqrt{qM}$ . The  $\ell_1$  norm can exceed the  $\ell_2$  norm by a factor of at most  $\sqrt{M}$ . As  $q \leq M$ , the proof follows. ■

We now prove Theorem 1.

**Proof.** The proof is by reduction from max cut, and uses for H2S the  $\ell_1$  maximization formulation.

Consider a graph  $G(V, E)$  with  $n$  vertices and  $m$  edges that serves as an input instance for max cut. Orient the edges of  $G$  arbitrarily. Our reduction uses an integer parameter  $M$  (setting  $M$  to be  $O(n^2 m^2)$  will suffice). We reduce the oriented  $G$  into an instance of H2S with  $k = Mn$  vectors of dimension  $d = Mm$  as follows.

The coordinates of vectors are partitioned into  $m$  blocks of  $M$  coordinates. Each block corresponds to one edge of  $G$ . Every vertex  $v_i$  of  $G$  gives rise to  $M$  vectors  $v_{i,1}, \dots, v_{i,M}$ . In each of these vectors, in every block  $B_e$  that corresponds to edge  $e$ :

1. If  $v_i$  is the head of  $e$  then all entries of  $B_e$  are  $+1$ .
2. If  $v_i$  is the tail of  $e$  then all entries of  $B_e$  are  $-1$ .
3. If  $v_i$  is not incident with  $e$ , then the entries of  $B_e$  in  $v_{i,j}$  (for  $1 \leq j \leq M$ ) are the  $j$ th codeword of the Hadamard code  $H_M$ .

**Yes instances.** Let  $(V_1, V_2)$  be the optimal cut for  $G$ , and suppose that it cuts  $c$  edges (necessarily  $c > \frac{m}{2}$ ). Consider the solution to the H2S instance that partitions the vectors into two clusters  $X_1$  and  $X_2$  in agreement with the partition  $(V_1, V_2)$ .

The value of the solution can be lower bounded as follows. There are  $Mn$  vectors and  $m$  blocks, each of size  $M$ . Consider a block that corresponds to an edge  $e$  that is cut. In each of  $X_1$  and  $X_2$  there is one endpoint of the edge, and this vertex has a monochromatic block that contributes  $M^2$  to the  $\ell_1$  norm. This might be partially offset by the other blocks. But the block of each vertex not incident with  $e$  can offset at most  $M^{3/2}$  of the  $\ell_1$  norm, by Proposition 2. As there are  $c$  edges in the cut, the value of the solution is at least  $c(2M^2 - (n-2)M^{3/2})$ . (The value is in fact higher because blocks corresponding to edges not in the cut also contribute to the  $\ell_1$  norm, but we ignore this further tightening of the parameters.)

**No instances.** Suppose now that no cut of  $G$  cuts  $c-1$  edges. Consider an arbitrary partition of the vectors of the H2S instance into two parts  $X_1$  and  $X_2$ . This partition corresponds to a fractional partition of  $G$ , where the extent  $x_i$  to which a vector  $v_i$  is in  $V_1$  is equal to the fraction of its vectors that are in  $X_1$ . Similarly,  $1-x_i$  is the extent to which  $v_i$  is in  $V_2$ . For an edge  $e = (v_i, v_j)$ , the extent to which it is cut is  $y_e = |x_i - x_j|$  (which of course equals  $|(1-x_i) - (1-x_j)|$ ).

For an arbitrary edge  $e$ , consider the contribution of the blocks associated with it to the  $\ell_1$  norm. The combination of two monochromatic blocks that are associated with its end points contribute  $M^2 y_e$  to the  $\ell_1$  norm of each of  $X_1$  and  $X_2$ . The Hadamard blocks associated with vertices that are not end points of  $e$  each contributes at most  $\sqrt{2}M^{3/2}$  towards the sum of  $\ell_1$  norms of  $X_1$  and  $X_2$ . (There is a multiplier of  $\sqrt{2}$  rather than just 1 because a vertex may be split among both sides of the cut. Proposition 2 allows one to upper bound the effect of this split by  $\sqrt{2}$ .) Summing up over all edges and all blocks, the value of any solution is at most  $2M^2 \sum_e y_e + \sqrt{2}(n-2)mM^{3/2}$ .

To bound  $\sum_e y_e$ , observe that local search can always change a fractional cut into an integer cut which is at least as large. Hence  $\sum_e y_e \leq c-1$ .

**Summary.** Subtracting the upper bound for *no* instances from the lower bound for *yes* instances, it follows that the *yes* instance leads to higher value than the *no* instance if  $2M^2 > (\sqrt{2}m + c)(n-2)M^{3/2}$ . Taking  $M > 2m^2 n^2$  suffices. ■

**Remark:** The value of  $M$  in the proof of Theorem 1 can be reduced to  $O(n^2)$  by using the fact that max-cut is APX-hard. By the results of [2], for *no* instances we may assume that there is no cut with  $0.942c$  edges.

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## References

- [1] Noga Alon, Benny Sudakov: On Two Segmentation Problems. *J. Algorithms* 33(1): 173–184 (1999).
- [2] Johan Hastad: Some optimal inapproximability results. *J. ACM* 48(4): 798–859 (2001).
- [3] Jon M. Kleinberg, Christos H. Papadimitriou, Prabhakar Raghavan: Segmentation Problems. *STOC* 1998: 473–482.
- [4] Jon M. Kleinberg, Christos H. Papadimitriou, Prabhakar Raghavan: Segmentation problems. *J. ACM* 51(2): 263–280 (2004).
- [5] Sharon Wulff, Ruth Uner, Shai Ben-David: Monochromatic Bi-Clustering. *ICML* (2) 2013: 145–153.